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STATIONARY AND QUASISTATIONARY SOLUTIONS OF THE COMPLEX GINZBURG–LANDAU EQUATION

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The structure of the phase space of stationary and quasistationary (moving at a constant velocity) solutions of the 1D complex Ginzburg–Landau equation is investigated by the methods of qualitative theory of ordinary differential equations. The existence of a countable set of double-loop heteroclinic trajectories is proved. The complex shock–hole–shock structures moving at a constant velocity along the coordinate correspond to these double-loop trajectories.

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1. Introduction

We consider the complex Ginzburg–Landau equation

$$\partial_t s = s - (1+i\beta)\,|s|^2 s + (1+i\alpha)\,\partial_x^2 s\,. \quad (1)$$

Equation (1) has a family of hole solutions in the form [1]

$$s_h(x-v_0t,t) = \Big[A_h(x-v_0t) + \eta v_0\Big] \times \exp\Big[i\Theta_h(x-v_0t) + ipv_0\,(x-v_0t) - i\Omega t\Big]\,, \quad (2)$$

where

$$A_h(\xi) = \lambda \tanh(k\xi)\,, \quad d\Theta_h/d\xi = \kappa \tanh(k\xi)\,.$$

λ, κ, and v_0 are constant coefficients, η is a complex one, and $p = 1/2\,(\alpha-\beta)$. The frequency Ω meets the dispersion relation

$$\Omega\,(Q,v_0) = \omega(Q) - v_0 Q\,, \quad (3)$$

$$\omega(Q) = \beta + (\alpha-\beta)\,Q^2.$$

As $\xi = x - v_0 t \to \pm\infty$, the solution (2) tends asymptotically to solutions in the form of plane waves

$$s_{ai}(\xi,t) = \sqrt{1-Q_i^2}\,\exp\Big(-i\Omega t + iQ_i\xi\Big), \quad i=1,2, \quad (4)$$

with asymptotic wavenumbers Q_1 (as $\xi\to-\infty$) and Q_2 (as $\xi\to+\infty$). Using dispersion relation (3), one readily finds that

$$\omega\,(Q_1) - v_0Q_1 = \omega\,(Q_2) - v_0Q_2\,. \quad (5)$$

Condition (5) is, actually, the condition of conservation (in a moving reference frame) of the constant phase difference between the asymptotic (as $\xi\to\pm\infty$) limits of solution (2).

2. Periodic solutions

Let us substitute the variables $s(x,t)=a(x,t)\,e^{-i\Omega t}$ into Eq. (1) and pass to the reference frame moving at the constant velocity v_0,

$$\partial_t a = v_0\,\partial_\xi a + (1+i\Omega)\,a - (1+i\beta)\,|a|^2 a + (1+i\alpha)\,\partial_\xi^2 a\,. \qquad (6)$$

Stationary solutions of the evolution problem (6) satisfy a set of ordinary differential equations in $\mathbf{R}^4$

$$\frac{da}{d\xi}=b\,,$$
$$\frac{db}{d\xi}=-\frac{1+i\Omega}{1+i\alpha}\,a+\frac{1+i\beta}{1+i\alpha}\,|a|^2 a-\frac{v_0}{1+i\alpha}\,b\,. \qquad (7)$$

Set (7) has two single-parametric families of solutions periodic in x

$$a_1(\xi)=A_{a1}e^{iQ_1\xi+i\varphi_1},\quad b_1(\xi)=iQ_1A_{a1}e^{iQ_1\xi+i\varphi_1}, \qquad (8)$$

$$a_2(\xi)=A_{a2}e^{iQ_2\xi+i\varphi_2},\quad b_2(\xi)=iQ_2A_{a2}e^{iQ_2\xi+i\varphi_2}, \qquad (9)$$

where $A_{ai}=\sqrt{1-Q_i^2}$, and the singular point is

$$a_0=0\,,\qquad b_0=0\,. \qquad (10)$$

3. Phase space structure in the case of $v_0=0$

Statement. *In the case of $v_0=0$, the set of Eqs. (7) is conservative and reversible (in the sense that x and $-x$ are interchangeable, that is $x\Rightarrow -x$). There exist two involutions*

$$a\Rightarrow -a\,, \qquad (11)$$

$$b\Rightarrow -b\,, \qquad (12)$$

that map the phase flux of Eqs. (7) into itself with the change $x\Rightarrow -x$. Besides, system (7) is invariant to the transform

$$a\Rightarrow ae^{i\varphi_0},\quad b\Rightarrow be^{i\varphi_0},\quad \varphi_0=\text{const.} \qquad (13)$$

Validity of the statement is readily verified by direct calculations. It is also easy to check that each of the above involutions maps the families of solutions (8) and (9) into each other. It follows immediately from this statement, that the phase flux is symmetrical with respect to the planes $a=0$ or $b=0$, with the substitution $x\Rightarrow -x$, and it is also invariant to a simultaneous turn by an angle φ_0 in the planes $a=0$ and $b=0$.

Let us rewrite (7) for $v_0=0$ in a slightly different form. Assuming that $a(x)=u(x)e^{i\varphi(x)}$ and using relation $\omega=\beta+(\alpha-\beta)\,Q^2$, we find

$$u'=v,$$
$$v'=u\left(\psi^2-Q^2\right)+B\left(u^2-A_a^2\right)u, \qquad (14)$$
$$u\psi'=-2v\psi+C\left(u^2-A_a^2\right)u,$$

where $\psi=d\varphi/dx$, $B=(1+\alpha\beta)/(1+\alpha^2)$, $C=(\beta-\alpha)/(1+\alpha^2)$. Set (14) has the dimension smaller than that of (7), but the latter contains a singularity at the plane $u=0$. The involutions

$$u\Rightarrow -u\,,\quad \psi\Rightarrow -\psi\,,\quad x\Rightarrow -x\,, \qquad (15)$$

$$v\Rightarrow -v\,,\quad \psi\Rightarrow -\psi\,,\quad x\Rightarrow -x\,, \qquad (16)$$

of system (14) correspond to involutions (11) and (12) of set (7).

The fixed points

$$u=A_a\,,\qquad \psi=Q\,, \qquad (17)$$

$$u=A_a\,,\qquad \psi=-Q\,, \qquad (18)$$

of set (14) correspond to periodic solutions (8) and (9) of system (7) at $\varphi_1=\varphi_2=0$. Note that the two other points

$$u=-A_a\,,\qquad \psi=Q\,, \qquad (19)$$

$$u=-A_a\,,\qquad \psi=-Q\,, \qquad (20)$$

correspond to same periodic solutions (8) and (9), but at $\varphi_1 = \varphi_2 = \pi$. Thus, when passing from (7) to (14), the system degenerates due to only two singular points (17), (19) or (18), (20) corresponding to each family of periodic solutions (8) or (9).

The types of singular points (17) and (18) were investigated in [2]. These points are saddles, but one of them has a 1D unstable S_2^u and a 2D stable S_2^s manifolds, while the other one vice versa. The type of the singular point on 2D manifolds is determined by parameters of the system. The spectrum of each singular point is purely real if

$$D = B_1^3 - \left[\frac{27}{2}\left(\frac{C_1}{A_1}\right)^2 + 6\right]B_1^2 + 12B_1 - 8 > 0\,, \quad (21)$$

and it is complex if $D<0$, where $B_1 = B\,(A_a/Q)^2$ and $C_1 = C\,(A_a/Q)^2$. Due to the symmetry of the phase flux of Eqs. (14) with respect to $u = \psi = 0$ and $v = \psi = 0$ (at $x \Rightarrow -x$), singular points (19) and (20) have a spectrum of eigenvalues coinciding with that of points (17) and (18).

The intersection of the 2D stable S_2^s and 2D unstable S_2^u manifolds of singular points (17), (18) and (19), (20) is structurally stable in $\mathbf{R}^3$. This means that the heteroclinic trajectory $\Gamma_s(\overline{\Gamma}_s)$ connecting equilibrium states (17) and (18) ((19) and (20)) is structurally stable in a general class of dynamical systems and, as was shown in [2], the shock solution

$$a_s(x) = A_s(x)\exp\left[i\theta_s(x) + i\varphi_s\right] \quad (22)$$

of basic equation (1) corresponds to this trajectory.

Consider now solution (2) of Eq. (1). Investigation of the hole solution (2), as $x \to \pm\infty$, verifies that, in the phase space of system (14), it corresponds to the heteroclinic trajectory $\Gamma_h(\overline{\Gamma}_h)$ connecting singular points (17) and (20) ((19) and (18)). One of the trajectories, belonging simultaneously to the 2D stable and 2D unstable manifolds of points (17)–(20), cannot be such a trajectory, because the hole solution is monotonic both as $x \to +\infty$ and $x \to -\infty$ for any values of the parameters. Thus, the only possibility is the intersection of 1D stable S_1^s and unstable S_1^u manifolds of equilibrium states (17) and (20) ((19) and (18)). In a general case, however, such an intersection is structurally unstable, moreover, it has codimension 2 in $\mathbf{R}^3$.

Let us turn again to system (7). It is clear from the above considerations for the stability points (17)–(20), that one of periodic solutions, (8) or (9), has a 2D stable, W_2^s, and a 3D unstable, W_3^u, manifolds, while the other solution vice versa. A shock solution in the phase space of system (7) corresponds to a structurally stable intersection of 3D stable and unstable manifolds of periodic solutions (8) and (9). The intersection of 2D manifolds W_2^s and W_2^u in $\mathbf{R}^4$ is, on the contrary, structurally unstable.

We recall that the phase flux of Eq. (7) is symmetrical (with $x \Rightarrow -x$) to the involution plane $a = 0$. Each of the 2D manifolds of periodic motions (8) and (9) generally intersects the plane $a = 0$ at the same point by virtue of this symmetry. On the other hand, the intersection of manifolds W_2^s and W_2^u at one point means that they intersect along the trajectory that connects periodic solutions (8) and (9). The situation described is structurally stable in the class of systems having only involutions (11) and (12). The invariance of phase flux to the rotation by an angle φ_0 in planes $a = 0$, $b = 0$, in turn, gives a whole family of heteroclinic trajectories instead of a single heteroclinic trajectory. This means that the intersection of plane $a = 0$ with manifolds W_2^s and W_2^u occurs not at one point but around a closed curve. Such a case, however, is no longer structurally stable in $\mathbf{R}^4$ and has the codimension one in the class of systems having two involutions (11) and (12) and invariant to transform (13).

It was assumed above that the parameters α, β, and Q of system (7) (or (14)) change independently of each other. In this case, the intersection of manifolds W_2^s and W_2^u (S_1^s and S_1^u) in the phase space of system (7) ((14)) is actually structurally unstable and is broken with a small perturbation of the parameters. However, the situation changes if we assume that only two parameters α and β are independent and choose the value of Q depending on α and β. It is clear that the dynamical systems possessing such a property form a film of codimension one.

Proposition 1: *For any values of the parameters α and β, giving real solution $Q(\alpha,\beta)$ to the equation*

$$Q^4\left(1-B-\frac{2}{9}C^2\right)+Q^2\left(B+\frac{4}{9}C^2\right)-\frac{2}{9}C^2=0\,, \tag{23}$$

there exists a heteroclinic trajectory Γ_h ($\overline{\Gamma}_h$) moving from point (17) ((19)) *to point* (20) ((18)) *and lying in the plane $P=\{u,v,\psi$: $\psi=(Q/A_a)u\}$ ($\overline{P}=\{u,v,\psi$: $\psi=(Q/A_a)u\}$).*

Proof: Let us consider the plane P that contains points (17) and (20). If the 1D manifolds S_1^s and S_1^u of these points belong completely to the plane P and intersect the axis $u=\psi=0$, they intersect each other and there exists a heteroclinic trajectory connecting points (17) and (20). This supposition follows immediately from the symmetry of phase flux (see (15)). The trajectories belonging completely to the plane P have no singularity in the plane $u=0$, as $\lim_{u\to 0}(\psi/u)=\text{const}$ for any curve lying in the plane P and intersecting the axis $u=\psi=0$.

Obviously, the phase trajectory belongs completely to the plane P only if the derivative of phase flux with respect to the normal to plane P equals zero at all points of this trajectory. The latter condition determines the curve

$$C=\left\{u,v,\psi:\ v=\frac{CA}{3Q}\left(u^2-A_a^2\right),\ \psi=\frac{Q}{A_a}u\right\}. \tag{24}$$

Therefore, the phase flux of system (14) is tangent to the plane P at the points lying on the curve C. Now it is necessary to verify that the curve C is really the solution of system (14) (phase trajectory). Direct substitution of (24) into (14) shows that the curve C is the solution of (14) for all values of Q complying with Eq. (23). The determinant of (23) is strictly positive and it is easy to check that the set of values α and β, giving the real solution of (23), is not empty. The proposition is proved.

Consequence: *Any phase trajectory of system* (14) *moving from the halfspace $u>0$ ($u<0$) to the halfspace $u<0$ ($u>0$) intersects the axis $u=\psi=0$.*

Note that the analytical expression of heteroclinic trajectory Γ_h (24) was derived in the proof of the proposition. The hole solution of basic equation (1) correspond to this trajectory.

4. Multiloop trajectories

We now consider the existence of multiloop heteroclinic trajectories, that is the trajectories corresponding to the solutions containing several hole and shock structures.

Definition: *The heteroclinic trajectories, belonging simultaneously to manifolds W_3^s and W_3^u of periodic solutions* (8) *and* (9) *and passing $(n-1)$ times near the heteroclinic trajectory Γ_h corresponding to the intersection of manifolds W_2^s and W_2^u, are called n-loop trajectories.*

According to this definition, a solution, containing n shocks and $(n-1)$ holes lying between the shocks, corresponds to the n-loop trajectory.

Proposition 2: *Let a heteroclinic contour exist in the phase space of system* (7)*, that is, two types of heteroclinic trajectories coexist to connect periodic solutions* (8) *and* (9)*. Then, a countable set of double-loop trajectories exists there.*

Proof: Apparently, a double-loop trajectory in the phase space of system (7) corresponds to an analogous trajectory in the space of system (14) and vice versa. It is easier to investigate these trajectories in $\mathbf{R}^3$, therefore we consider the latter space. We suppose that fixed points (17)–(20) are saddle foci and (17) has 2D stable and 1D unstable manifolds. Then, the shock solution corresponds to a structurally stable heteroclinic trajectory Γ_s, along which the unstable S_2^u and stable S_2^S manifolds of points (17), (18) intersect. By virtue of the phase space symmetry with respect to $u=\psi=0$, an analogous intersection (trajectory $\overline{\Gamma}_s$) occurs for the manifolds $\overline{S}_2^u$ and $\overline{S}_2^s$ of singular points (19), (20).

The simplest double-loop heteroclinic trajectory $\overline{\Gamma}_{shs}$ originates at singular point (20), moves along the unstable manifold $\overline{S}_2^u$ near the trajectory $\overline{\Gamma}_s$, enters the vicinity of point (19), then passes along the curve $\overline{\Gamma}_h$ to the vicinity of point (18), and, finally, moving along Γ_s it ends at point (17). Clearly, one more double-loop trajectory Γ_{shs} also exists, that is symmetrical to $\overline{\Gamma}_{shs}$ and connects points (18) and (19).

Let us designate as T_h a global map of the secant in the neighborhood of heteroclinic trajectory Γ_h and, as T_0, a local map in the neighborhood of saddle focus.

We consider singular point (19). System (14) may be linearized in the vicinity of saddle focus and, offer a certain transform of coordinates, it can be converted into

$$\dot{\overline{z}} = \lambda\overline{z},\ \lambda > 0, \quad \dot{\overline{\varphi}} = \omega,\ \omega > 0, \quad \dot{\overline{\rho}} = -\overline{\rho}, \tag{25}$$

where $(\overline{z}, \overline{\psi}, \overline{\rho})$ are the variables in the local cylindrical reference frame in the neighborhood of point (19). Consider the set $\overline{R} = \{(\overline{z}, \overline{\psi}, \overline{\rho})\colon \overline{z} \in (0, \overline{z}_1],\ \overline{\varphi} = \overline{\varphi}_0,\ \overline{\rho} = \overline{\rho}_0\}$, lying in the vicinity of saddle focus (19). We take the secant $\overline{Z} = \{\overline{z}\colon \overline{z} = \overline{z}_1\}$ and find the set $T_0\overline{R}$ on it.

From the first equation in system (25) we have $\overline{z} = \overline{z}_1 = \overline{z}e^{\lambda t}$, $\overline{z} \in (0, \overline{z}_1]$, and then

$$\overline{\varphi}^{(1)} = \overline{\varphi}_0 + \frac{\omega}{\lambda} \ln \frac{\overline{z}_1}{\overline{z}}, \qquad \overline{\rho}^{(1)} = \overline{\rho}_0 \left(\frac{\overline{z}}{\overline{z}_1}\right)^{1/\lambda}. \tag{26}$$

Apparently, Eqs. (26) describe a spiral connecting the points $\overline{\varphi} = \overline{\varphi}_0$, $\overline{\rho} = \overline{\rho}_0$, and $\overline{\rho} = 0$.

Let us take the cylindrical surface $\overline{K} = \{\overline{z}, \overline{\varphi}, \overline{\rho}\colon\ \overline{z} \in (0, \overline{z}_1],\ \overline{\varphi} \in [0, 2\pi),\ \overline{\rho} = \overline{\rho}_0\}$ in the vicinity of singular point (19). The intersection of 2D manifold $\overline{S}_2^u$ of singular point (20) with the surface $\overline{K}$ occurs along the curve $\overline{C}$ originating from the point $\overline{z} = 0$, $\overline{\rho} = \overline{\rho}_0$, $\overline{\varphi} = \overline{\varphi}_0$ on the trajectory $\overline{\Gamma}_s$ directed along $\overline{K}$ towards larger values of $\overline{z}$ [1]. The curve $\overline{C} \in \overline{K}$ is topologically equivalent to the section $\overline{R} \in \overline{K}$. Consequently, its image on the secant $\overline{Z}$ is also a spiral moving to the point $\overline{\rho} = 0$. Thus, the spiral $\overline{G}$ originating from the point $\overline{\gamma} = \{(\overline{z}, \overline{\varphi}, \overline{\rho})\colon \overline{z} = \overline{z}_1, \overline{\rho} = 0\}$ is the image of element $\overline{D}$ ($\overline{D} \cap \overline{\Gamma}_s \neq 0$) of the manifold $\overline{S}_2^u$ on the secant $\overline{Z} = \{\overline{z}\colon \overline{z} = \overline{z}_1\}$ in the vicinity of point (19).

[1] We analyze the half-space $\overline{z} \geqslant 0$.

Repeating the same procedure, we can show that the spiral G originating from the point $\gamma = \{(z, \varphi, \rho)\colon z = z_1,\ \rho = 0\}$ is the image of element D ($D \cap \Gamma_s \neq 0$) of the manifold S_2^s of singular point (17) on the secant $Z = \{z\colon z = z_1\}$, where (z, φ, ρ) are the local coordinates in the neighborhood of saddle focus (18).

The global map T_h transforms the point $\overline{\gamma}$ to γ, and the spiral $\overline{G}$ to $\overline{G}^{(1)}$ with the center at the point γ. One readily verifies that the spirals G and $\overline{G}^{(1)}$ rotate in opposite directions due to the phase space symmetry. Therefore, they have a countable number of intersections. Each intersection corresponds to a double-loop heteroclinic trajectory connecting points (20) and (17). Thus, the proposition is proved.

Note that the centers of spirals G and $\overline{G}^{(1)}$ do not coincide in the absence of heteroclinic trajectory Γ_h. However, if the manifolds S_1^s and S_1^u of singular points (19), (18) are sufficiently close to each other, the spirals intersect, although the number of intersections in now finite rather than countable.

It is clear that more complicated n-loop ($n > 2$) heteroclinic trajectories also may exist. But the proof of this statement is much more complicated than that presented above. Therefore, numerical investigation of the phase space of system (14) is of great interest. A heteroclinic trajectory corresponding to one of the shock structures was found numerically in [2].

5. Phase space structure in the case of $v_0 \neq 0$

For $v_0 \neq 0$, the set of equations (7) is not reversible, like in the case of $v_0 = 0$. But the phase flux is yet invariant to transform (13).

Let us suppose, as before, that $a(x) = u(x)e^{i\varphi(x)}$ and transform system (7) into

$$u' = v\,,$$

$$v' = u\left(\psi^2 - Q_1^2\right) + B\left(u^2 - A_{a1}^2\right)u + \alpha v_0 D\left(Q_1 - \psi\right)u - v_0 D v\,,$$
$$u\psi' = -2v\psi + C\left(u^2 - A_{a1}^2\right)u + v_0 D\left(Q_1 - \psi\right)u + \alpha v_0 D v\,, \tag{27}$$

where $\psi = d\varphi/dx$, $D = (1+\alpha^2)^{-1}$, and $A_{ai} = \sqrt{1-Q_i^2}$. System (27) has the fixed points

$$u = A_1\,, \qquad \psi = Q_1\,, \tag{28}$$
$$u = -A_1\,, \qquad \psi = Q_1\,, \tag{29}$$
$$u = A_2\,, \qquad \psi = Q_2\,, \tag{30}$$
$$u = -A_2\,, \qquad \psi = Q_2\,, \tag{31}$$

and the straight line $v = u = 0$. Note that the asymptotic wavenumbers Q_1 and Q_2 are related as $Q_2 + Q_1 = v_0/(\alpha - \beta)$. Consequently, as $v_0 \to 0$, equilibrium states (28)–(31) transform to (17)–(20). System (27) is degenerate relative to (7). The condition of invariance (13) reduces for system (27) to the condition of phase flux invariance to the transforms $u \Rightarrow -u$ and $v \Rightarrow -v$.

The stability of singular points (28)–(31) can be found directly from analysis of the linearized system (27). However, due to a continuous dependence of the roots of algebraic equation on its coefficients, we state that the character and stability of singular points (28)–(31) and (17)–(20) coincide at sufficiently small v_0. If, at $v_0 = 0$, a structurally stable heteroclinic trajectory exists to connect points (17) and (18), (19) and (20), then, at small enough v_0, a structurally stable heteroclinic trajectory exists there, corresponding to the intersection of 2D stable and unstable manifolds of singular points (28) and (30) ((29) and (31)). A shock solution of the initial evolution equation (1), moving at the constant velocity v_0 along the x-axis, corresponds to this trajectory. The intersection of 1D stable and unstable manifolds of singular points (28) and (31) ((29) and (30)) corresponds to the hole solutions moving at constant velocity v_0 in the phase space of system (27). Like in the case of $v_0 = 0$, the heteroclinic trajectory corresponding to this intersection is structurally unstable in the general class of dynamical systems. However, the problem disappears if we assume that one of the parameters of system (27), that is Q_1, depends on the other ones.

Thus, analysis of the phase space of system (27) indicates that, besides fixed (immovable) structures, moving hole and shock structures must exist there. Moreover, Proposition 2 on multiloop trajectories is completely valid for the case of $v_0 \neq 0$. Indeed, its proof is based on the existence of structurally stable heteroclinic trajectories connecting the singular points and on the local analysis of phase flux in the neighborhood of the saddle focus. Complex shock–hole–shock structures moving at the constant velocity v_0 correspond to such trajectories as solutions to the original evolution problem.

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