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The diversity of steady state solutions of the complex Ginzburg–Landau equation

M. Bazhenov [a], T. Bohr [b], K. Gorshkov [a], M. Rabinovich [a,c]

[a] *Institute of Applied Physics, 46 Uljanov Street, 603600 Nizhny Novgorod, Russian Federation*

[b] *Center for Chaos and Turbulence Studies, The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen, Denmark*

[c] *Institute for Nonlinear Science, UCSD, La Jolla, CA 92093-0402, USA*

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Abstract

The structure of the phase space of stationary and quasi-stationary (i.e., uniformly translating) solutions of 1D CGLE is investigated by methods of the qualitative theory of ordinary differential equations. The Nozaki–Bekki holes are seen as heteroclinic connections which are made structurally stable by an involution symmetry in phase space. The existence of a countable set of double-loop heteroclinic trajectories is proved, which corresponds to complex "shock–hole–shock" structures both motionless and moving with constant velocity v_0 along the x-axis.

1. Introduction

We consider the complex Ginzburg–Landau equation

$$\partial_t s = s - (1 + \mathrm{i}\beta)|s|^2 s + (1 + \mathrm{i}\alpha)\partial_x^2 s, \qquad (1)$$

which describes the spatio-temporal evolution of a 1D extended system near the Hopf bifurcation point. Eq. (1) has a family of hole solutions in the form [1]

$$s_{\mathrm{h}}(x - v_0 t, t) = [A_{\mathrm{h}}(x - v_0 t) + \eta v_0] \times \exp[\mathrm{i}\Theta_{\mathrm{h}}(x - v_0 t) + \mathrm{i}p v_0 (x - v_0 t) - \mathrm{i}\Omega t], \qquad (2)$$

where

$$A_{\mathrm{h}}(\xi) = \lambda \tanh(k\xi),$$

$$\mathrm{d}\Theta_{\mathrm{h}}/\mathrm{d}\xi = \kappa \tanh(k\xi).$$

Here λ, κ, and v_0 are real constants, η is a complex constant, and $p = 1/2(\alpha - \beta)$. For $\xi = x - v_0 t \to \pm\infty$, the solution (2) tends asymptotically to solutions in the form of plane waves,

$$s_{ai}(\xi, t) = \sqrt{1 - Q_i^2}\, \mathrm{e}^{-\mathrm{i}\Omega t + \mathrm{i}Q_i \xi}, \quad i = 1, 2, \qquad (3)$$

with asymptotic wave numbers Q_1 (for $\xi \to -\infty$) and Q_2 (for $\xi \to +\infty$). The frequency Ω satisfies the dispersion relations

$$\Omega = \omega(Q_i) - v_0 Q_i,$$

$$\omega(Q_i) = \beta + (\alpha - \beta) Q_i^2. \qquad (4)$$

From (4) one can readily find

$$\omega(Q_1) - v_0 Q_1 = \omega(Q_2) - v_0 Q_2. \qquad (5)$$

Condition (5) is, actually, the condition of the conservation (in a moving reference frame) of the constant phase difference between the asymptotic (for $\xi \to \pm\infty$) limits of the solution (2).

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Eq. (2) describes a structure moving along the x-axis with constant velocity v_0. For $v_0 = 0$, (2) takes a very simple form,

$$s_h(x,t) = A_h(x)e^{i\Theta_h(x) - i\omega t}, \tag{6}$$

where

$$A_h(x) = \sqrt{1-Q^2}\tanh(kx), \tag{7}$$

$$\frac{d\Theta_h}{dx} = -Q\tanh(kx). \tag{8}$$

The structures (2) and (6) are sources of plane waves [2]. Another type of solution to (1) we will be interested in is a shock solution. It is a sink that is formed in the interaction of two opposing incident plane waves. The solution corresponding to a fixed (motionless) shock may be written as

$$s_s(x,t) = A_s(x)e^{i\Theta_s(x) - i\omega t}. \tag{9}$$

Here $A_s(x) > 0$, $A_s(x) \to \sqrt{1-Q^2}$, $\Theta_s(x) \to \pm Qx$ for $x \to \pm\infty$.

Fig. 1 shows the field amplitude distribution typical of Eq. (1). The distribution contains a sequence of alternating holes (sources) and shocks (sinks).

Let us make a substitution of variables $s(x,t) = a(x,t)e^{-i\Omega t}$ in Eq. (1) and pass to the reference frame moving with constant velocity v_0. We obtain

$$\partial_t a = v_0\partial_\xi a + (1+i\Omega)a - (1+i\beta)|a|^2 a + (1+i\alpha)\partial_\xi^2 a. \tag{10}$$

Stationary solutions of the evolution problem (10) satisfy a system of ordinary differential equations in $\mathbb{R}^4$:

$$\frac{da}{d\xi} = b,$$

$$\frac{db}{d\xi} = -\frac{1+i\Omega}{1+i\alpha}a + \frac{1+i\beta}{1+i\alpha}|a|^2 a - \frac{v_0}{1+i\alpha}b. \tag{11}$$

System (11) will be analysed in detail in this paper. The phase space structure for $v_0 = 0$ will be considered in Section 2. We will show which phase trajectories correspond to fixed hole and shock structures and we will discuss their structural stability. In Section 3 we will prove the existence of multiloop heteroclinic trajectories corresponding to complex structures which consist of a sequence of holes and shocks. Finally, the results obtained will be generalized to the case $v_0 \neq 0$ in Section 4.

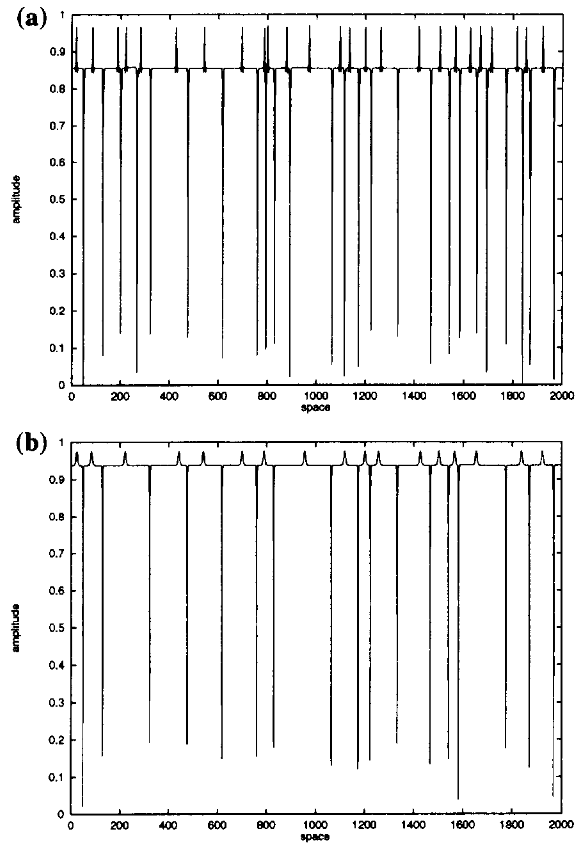

Fig. 1. Snapshot of the field amplitude $|s(x,t_0)|$ of Eq. (1) for $\alpha = 0, \beta = 1.5$ (a) and $\alpha = 0.6, \beta = 2$ (b) in the region $\{-L \leqslant x \leqslant L\}, L \gg 1$ with periodic boundary conditions. Motionless structures, i.e. holes (sources) and shocks (sinks), alternate.

2. Stationary structures

Let us take $v_0 = 0$ in (11) and consider a system of equations

$$\frac{da}{dx} = b, \quad \frac{db}{dx} = -\frac{1+i\Omega}{1+i\alpha}a + \frac{1+i\beta}{1+i\alpha}|a|^2 a. \tag{12}$$

Note that the case $v_0 = 0$ is the physically most interesting case. It was shown in Ref. [3] (see also Ref. [4]) that, when the right-hand side of (1) is disturbed by a term of the form $d|s|^4 s$, the family of hole solutions (2) breaks down, and only stationary structures with $v_0 = 0$ are retained.

System (12) has two one-parametric families of solutions that are periodic along x,

$$a_1(x) = A_a e^{iQx+i\varphi_1}, \quad b_1(\xi) = iQA_a e^{iQx+i\varphi_1}, \quad (13)$$

$$a_2(x) = A_a e^{-iQx+i\varphi_2}, \quad b_2(\xi) = -iQA_a e^{-iQx+i\varphi_2}, \quad (14)$$

where $A = \sqrt{1-Q^2}$ and φ_i is an arbitrary constant.

The system of equations (12) is conservative and reversible (in the sense that x and $-x$ are interchangeable, i.e., $x \Rightarrow -x$). There exist two involutions,

$$a \Rightarrow -a \quad (15)$$

and

$$b \Rightarrow -b, \quad (16)$$

which map the phase flow of Eq. (11) into itself with the change $x \Rightarrow -x$. Besides, the system (11) is invariant under the transformation

$$a \Rightarrow a e^{i\varphi_0}, \quad b \Rightarrow b e^{i\varphi_0}, \quad \varphi_0 = \text{const}. \quad (17)$$

The validity of these last statements is readily verified by direct calculations. It is also easy to check that each of the above mentioned involutions maps the families of solutions (13) and (14) into each other. This follows immediately from the statement that the phase flow is symmetric with respect to the plane $a = 0$ or $b = 0$, with the change $x \Rightarrow -x$ taken into account, and invariant to a simultaneous turn by an angle φ_0 in the planes $a = 0$ and $b = 0$.

Let us rewrite (12) for $v_0 = 0$ in a slightly different form. Supposing $a(x) = u(x)e^{i\varphi(x)}$ and using the relation $\omega = \beta + (\alpha - \beta)Q^2$ we find

$$u' = v, \quad v' = u(\psi^2 - Q^2) + B(u^2 - A_a^2)u,$$
$$u\psi' = -2v\psi + C(u^2 - A_a^2)u, \quad (18)$$

where $\psi = d\varphi/dx$, $B = (1+\alpha\beta)/(1+\alpha^2)$, $C = (\beta - \alpha)/(1+\alpha^2)$.

Note that we do not demand the function $u(x)$ to be positive. Consequently, the functions $u(x)$ and $\varphi(x)$ coincide with the amplitude and phase, respectively, of the complex variable $a(x)$ only if $u(x) \geqslant 0$. For $u(x) < 0$, the variable $\varphi(x)$ differs from phase by π.

System (18) has a dimension smaller than that of (11) but it contains a singularity in the plane $u = 0$. The involutions

$$u \Rightarrow -u, \quad \psi \Rightarrow -\psi, \quad x \Rightarrow -x, \quad (19)$$

$$v \Rightarrow -v, \quad \psi \Rightarrow -\psi, \quad x \Rightarrow -x, \quad (20)$$

of system (18) correspond to involutions (15) and (16) of system (11).

The fixed points

$$P_1 = \{u = A_a,\ \psi = Q,\ v = 0\}, \quad (21)$$

$$P_2 = \{u = A_a,\ \psi = -Q,\ v = 0\}, \quad (22)$$

of system (18) correspond to the periodic solutions (13) and (14) of system (11) for $\varphi_1 = \varphi_2 = 0$. Note that the two other points,

$$\overline{P_1} = \{u = -A_a,\ \psi = Q,\ v = 0\}, \quad (23)$$

$$\overline{P_2} = \{u = -A_a,\ \psi = -Q,\ v = 0\}, \quad (24)$$

correspond to the same periodic solutions (13) and (14) but with $\varphi_1 = \varphi_2 = \pi$. Thus, in passing from (11) to (18), a degeneracy appears due to the fact that the family of periodic solutions (13) ((14)) now only corresponds to the singular points P_1 and $\overline{P_1}$ (P_2 and $\overline{P_2}$).

The types of the singular points P_1 and P_2 were investigated in Ref. [5]. Both these points are saddles, but P_1 has a 1D unstable S_1^u and a 2D stable S_2^s manifolds, while P_2 vice versa. The type of the singular point on 2D manifolds is determined by the parameters of the system. The spectrum of each singular point is purely real if

$$D = B_1^3 - \left[\frac{27}{2}\left(\frac{C_1}{A_1}\right)^2 + 6\right]B_1^2 + 12B_1 - 8 > 0, \quad (25)$$

and complex if $D < 0$, where $B_1 = B(A_a/Q)^2$, $C_1 = C(A_a/Q)^2$. Due to the symmetry of the phase flow of Eq. (18) with respect to $u = \psi = 0$ and $v = \psi = 0$ (with $x \Rightarrow -x$), the singular points $\overline{P_1}$, $\overline{P_2}$ have a spectrum of eigenvalues coinciding with the spectrum of the points P_1 and P_2, respectively.

Consider now in more detail the hole and shock solutions of Eq. (1). As mentioned above, a shock solution (9) connects two plane waves incident towards each other from $\pm\infty$. A structurally stable intersection of the 2D stable S_2^s and 2D unstable S_2^u manifolds of the singular points P_1 and P_2 corresponds to this solution in the phase space of system (18) [5]. Structural stability of the heteroclinic trajectory Γ_s connecting the equilibrium states P_1 and P_2 indicates that the shock is retained at arbitrary disturbance of the right-hand side of system (18), in particular, at arbitrary change of the parameters α, β, and Q. The latter circumstance demonstrates that, being a sink, the shock is a "nonself-sustained" structure; instead, it depends on the behaviour of the field at infinity. It is clear that in addition to Γ_s there exists one more heteroclinic trajectory $\overline{\Gamma_s}$ connecting the points $\overline{P_1}$ and $\overline{P_2}$.

The situation is different for the solution (2) of Eq. (1). Investigation of the hole solution (2) for $x \to \pm\infty$ verifies that a hole solution in the phase space of system (18) corresponds to a heteroclinic trajectory Γ_h ($\overline{\Gamma}_h$) connecting the singular points P_1 and $\overline{P_2}$ ($\overline{P_1}$ and P_2). One of the trajectories belonging simultaneously to the 2D stable and 2D unstable manifolds of the points (21)–(24) cannot be such a trajectory because the hole solution is monotonic both for $x \to +\infty$ and $x \to -\infty$ for any value of the parameters and contains a point at which the field amplitude vanishes. Thus, the only possibility is the intersection of 1D stable and unstable manifolds of the equilibrium states P_1 and $\overline{P_2}$ ($\overline{P_1}$ and P_2). In the general case, however, such an intersection is structurally unstable; moreover, it has co-dimension two in $\mathbb{R}^3$. The question is: why, in spite of this circumstance, do hole structures exist in a broad interval of the parameters α and β?

Proposition 1. The intersection of 1D manifolds S_1^u and $\overline{S_1^s}$ (S_1^s and $\overline{S_1^u}$) of the singular points P_1 and $\overline{P_2}$ ($\overline{P_1}$ and P_2), in the class of dynamical systems possessing the involutions (19), (20), has co-dimension 1.

Proof. Let the 1D separatrix S_1^u of the singular point P_1 intersect the line $\psi = u = 0$ at the point a. Such an intersection is structurally unstable because it breaks down when the right-hand side of system (18) is perturbed and has co-dimension 1. However, the line $\psi = u = 0$ is the involution axis; consequently, the separatrix $\overline{S_1^s}$ of the point $\overline{P_2}$ intersects the axis $\psi = u = 0$ at the point a too. Thus, the heteroclinic trajectory connecting the points P_1 and $\overline{P_2}$ has co-dimension 1. The same is true for the heteroclinic trajectory corresponding to the intersection of the one-dimensional manifolds S_1^s and $\overline{S_1^u}$ of the singular points P_2 and $\overline{P_1}$, respectively. The proposition is proved.

It was assumed above that the parameters α, β and Q of system (11) (or (18)) can change independent of each other. In this case, the intersection of the manifolds S_1^u and $\overline{S_1^s}$ (S_1^s and $\overline{S_1^u}$) in the phase space of system (18) is really structurally unstable and disappears when the parameters are perturbed. However, the situation will change if it is assumed that only two parameters, e. g. α and β, are independent and that the value of Q depends on α and β. It is clear that the dynamical systems possessing such a property will form a "film" of co-dimension 1.

Proposition 2. For any value of the parameters α and β giving the real solution $Q(\alpha, \beta)$ of the equation

$$Q^4(1 - B - \tfrac{2}{9}C^2) + Q^2(B + \tfrac{4}{9}C^2) - \tfrac{2}{9}C^2 = 0, \quad (26)$$

there exists a heteroclinic trajectory Γ_h ($\overline{\Gamma}_h$) moving from the point P_1 ($\overline{P_1}$) to the point $\overline{P_2}$ (P_2) and lying in the plane $L = \{u, v, \psi \; : \; \psi = (Q/A_a)u\}$ ($\overline{L} = \{u, v, \psi \; : \; \psi = (Q/A_a)u\}$).

Proof. Let us consider the plane L which contains the points P_1 and $\overline{P_2}$. One must note that the trajectories which belong completely to the plane L have no singularity on the line $u = \psi = 0$ as $\lim_{u \to 0}(\psi/u) = \text{const}$ for any curve lying in the plane L and intersecting the axis $u = \psi = 0$. Obviously, the phase trajectory will belong completely to the plane L only if the derivative of the phase flow with respect to the normal to the plane P equals zero at all points of this trajectory. The latter condition determines the curve

$$C = \left\{ u, v, \psi \; : \; v = \frac{CA}{3Q}(u^2 - A_a{}^2), \psi = \frac{Q}{A_a}u \right\}. \quad (27)$$

So, the phase flow of system (18) is tangent to the plane L at the points lying on the curve C. Now let us verify that the curve C is really the solution of system (18) (C is the phase trajectory). Direct substitution of (27) into (18) shows that the curve C is the solution of system (18) for all values of Q satisfying (26). The determinant of (26) is strictly positive and it is easy to check that the set of values α and β giving a real solution of Eq. (26) is not empty. Thus the proposition is proved.

Note that the analytical expression for the heteroclinic trajectory Γ_h (Eq. (27)) was obtained in the proof of the proposition. The hole solution of the basic equation (1), first obtained in Ref. [1] by Hirota's bilinear method [6], corresponds to this trajectory.

Thus, the symmetry of phase space and the dependence of the parameter Q on α and β explain the fact that the heteroclinic trajectories corresponding to hole structures are retained at arbitrary values of α and β in Eq. (1).

3. Multiloop trajectories

We now consider the existence of multiloop heteroclinic trajectories, i.e., the trajectories to which the solutions containing several hole and shock structures correspond.

Definition. The heteroclinic trajectories belonging simultaneously to the 2D stable manifold of the fixed point P_1 ($\overline{P_1}$) and the 2D unstable manifold of the fixed point P_2 ($\overline{P_2}$) and passing near the heteroclinic trajectory $\overline{\Gamma_h}$ (Γ_h) we will call double-loop trajectories.

According to this definition the solution containing two shocks and one hole lying between them corresponds to a double-loop trajectory.

Proposition 3. Let there co-exist in the phase space of system (18) both types of heteroclinic trajectories considered above. The first of them corresponds to the intersections of 1D stable and unstable manifolds of the singular points P_1 and $\overline{P_2}$ (P_2 and $\overline{P_1}$), and the second one to the intersections of 2D stable and unstable manifolds of the points P_1 and P_2 ($\overline{P_1}$ and $\overline{P_2}$). Then there exists a countable set of double-loop trajectories.

Proof. Suppose that the fixed points (21)–(24) are saddle-foci and P_1 has a two-dimensional stable and a one-dimensional unstable manifolds. Then there corresponds to a shock solution a structurally stable heteroclinic trajectory Γ_s along which the unstable S_2^u and stable S_2^s manifolds of the points P_1 and P_2 intersect. By virtue of the phase space symmetry with respect to $u = \psi = 0$, an analogous intersection (trajectory $\overline{\Gamma}_s$) occurs for the manifolds $\overline{S}_2^u$ and $\overline{S}_2^s$ of the singular points $\overline{P_1}$ and $\overline{P_2}$.

The simplest double-loop heteroclinic trajectory $\overline{\Gamma}_{shs}$ originates at the singular point $\overline{P_2}$, moves along the unstable manifold $\overline{S}_2^u$ near the trajectory $\overline{\Gamma}_s$, enters the neighbourhood of the point $\overline{P_1}$, then passes along the curve $\overline{\Gamma}_h$ to the neighbourhood of the point P_2, and, finally, moving along Γ_s it ends at the point P_1. Clearly, there exists an additional double-loop trajectory Γ_{shs} which connects the points P_2 and $\overline{P_1}$.

Let us designate by T_h a global map of a secant in the neighbourhood of the heteroclinic trajectory Γ_h and by T_0 a local map in the neighbourhood of saddle-focus.

Consider the singular point $\overline{P_1}$. System (18) may be linearized in the neighbourhood of the saddle-focus and, by a coordinate transformation, it can be written as

$$\dot{\overline{z}} = \lambda \overline{z}, \quad \lambda > 0, \quad \dot{\overline{\varphi}} = \omega,$$
$$\omega > 0, \quad \dot{\overline{\rho}} = -\overline{\rho}, \tag{28}$$

where $(\overline{z}, \overline{\psi}, \overline{\rho})$ are the variables in the local cylindrical reference frame in the neighbourhood of the point $\overline{P_1}$. Consider the set $\overline{R} = \{(\overline{z}, \overline{\psi}, \overline{\rho}) \; : \; \overline{z} \in (0, \overline{z}_1], \overline{\varphi} = \overline{\varphi}_0, \overline{\rho} = \overline{\rho}_0\}$ lying in the neighbourhood of the saddle-focus $\overline{P_1}$. We take the secant $\overline{Z} = Z = \{\overline{z} : \overline{z} = \overline{z}_1\}$ and find the set $T_0\overline{R}$ on it.

From the first equation in system (28) we have $\overline{z} = \overline{z}_1 = \overline{z} e^{\lambda t}$, $\overline{z} \in (0, \overline{z}_1]$. Then

$$\overline{\varphi}^{(1)} = \overline{\varphi}_0 + \frac{\omega}{\lambda} \ln \frac{\overline{z}_1}{\overline{z}},$$
$$\overline{\rho}^{(1)} = \overline{\rho}_0 (\overline{z}/\overline{z}_1)^{1/\lambda}. \tag{29}$$

Apparently, Eqs. (29) describe the spiral connecting the points $\overline{\varphi} = \overline{\varphi}_0$, $\overline{\rho} = \overline{\rho}_0$ and $\overline{\rho} = 0$.

Consider the cylindrical surface $\overline{K} = K = \{\overline{z}, \overline{\varphi}, \overline{\rho} : \overline{z} \in (0, \overline{z}_1], \overline{\varphi} \in [0, 2\pi), \overline{\rho} = \overline{\rho}_0\}$ in the neighbourhood of the singular point $\overline{P_1}$. The intersection of the two-dimensional manifold $\overline{S}_2^u$ of the singular point $\overline{P_2}$ with the surface $\overline{K}$ occurs along the curve $\overline{C}$ originating from the point $\overline{z} = 0, \overline{\rho} = \overline{\rho}_0, \overline{\varphi} = \overline{\varphi}_0$ on the trajectory $\overline{\Gamma}_s$ and is directed along $\overline{K}$ towards larger values of the variable $\overline{z}$. [1] The curve $\overline{C} \in \overline{K}$ is topologically

[1] We analyze the half-space $\overline{z} \geqslant 0$.

equivalent to the section $\overline{R} \in \overline{K}$. Consequently, its image on the secant $\overline{Z}$ is also a spiral moving to the point $\overline{\rho} = 0$.

Thus, the spiral $\overline{G}$ originating from the point $\overline{\gamma} = \{(\overline{z}, \overline{\varphi}, \overline{\rho}) \ : \ \overline{z} = \overline{z}_1, \overline{\rho} = 0\}$ is the image of the element $\overline{D}$ ($\overline{D} \cap \overline{\Gamma}_s \neq 0$) of the manifold $\overline{S_2^u}$ on the secant $\overline{Z} = \{\overline{z} : \ \overline{z} = \overline{z}_1\}$ in the neighbourhood of the point P_2.

Repeating the same procedure we can show that the spiral G originating from the point $\gamma = \{(z, \varphi, \rho) \ : \ z = z_1, \rho = 0\}$ is the prototype of the element D ($D \cap \Gamma_s \neq 0$) of the manifold S_2^s of the singular point P_1 on the secant $Z = \{z \ : \ z = z_1\}$, where (z, φ, ρ) are the local coordinates in the neighbourhood of the saddle-focus P_2.

The global map T_h transforms the point $\overline{\gamma}$ to γ, and the spiral $\overline{G}$ to $\overline{G}^{(1)}$ with the center at the point γ. One can readily verify that the spirals G and $\overline{G}^{(1)}$ rotate in opposite directions due to phase space symmetry. Consequently, they have a countable number of intersections. Each intersection corresponds to a double-loop heteroclinic trajectory connecting the points $\overline{P_2}$ and P_1. Different double-loop heteroclinic trajectories correspond to the shock–hole–shock structures that are slightly different in shape. Thus, the proposition is proved.

Note that the centers of the spirals G and $\overline{G}^{(1)}$ do not coincide in the absence of the heteroclinic trajectory Γ_h. However, if the manifolds S_1^s and $\overline{S_1^u}$ of the singular points P_2, $\overline{P_1}$ are sufficiently close to each other, the spirals intersect, although the number of intersections is now finite rather than countable.

It is clear that more complicated n-loop ($n > 2$) heteroclinic trajectories may exist too. The proof of this statement is more complicated than the proof of Proposition 3. However, such a proof may be given by analogy with that presented above.

4. Nonstationary structures ($v_0 \neq 0$)

For $v_0 \neq 0$, the system of equations (11) is not reversible, like in the case $v_0 = 0$. But the phase flow is also invariant to the transformation (17).

We suppose, as before, that $a(x) = u(x)e^{i\varphi(x)}$ and transform system (11) into the form

$$\begin{aligned}
u' &= v,\\
v' &= u(\psi^2 - Q_1^2) + B(u^2 - A_{a1}^2)u\\
&\quad + \alpha v_0 D(Q_1 - \psi)u - v_0 D v,\\
u\psi' &= -2v\psi + C(u^2 - A_{a1}^2)u\\
&\quad + v_0 D(Q_1 - \psi)u + \alpha v_0 D v,
\end{aligned} \tag{30}$$

where $\psi = d\varphi/dx$, $D = 1/(1+\alpha^2)$, and $A_{ai} = \sqrt{1 - Q_i^2}$. System (30) has the following fixed points,

$$P_1' = \{u = A_1,\ \psi = Q_1,\ v = 0\}, \tag{31}$$

$$P_2' = \{u = A_2,\ \psi = Q_2,\ v = 0\}, \tag{32}$$

$$\overline{P_1'} = \{u = -A_1,\ \psi = Q_1,\ v = 0\}, \tag{33}$$

$$\overline{P_2'} = \{u = -A_2,\ \psi = Q_2,\ v = 0\}. \tag{34}$$

Note that the asymptotic wave numbers Q_1 and Q_2 are related by $Q_2 + Q_1 = v_0/(\alpha - \beta)$. Consequently, for $v_0 \to 0$, the equilibrium states (31)–(34) transform to (21)–(24). The condition of invariance (17) reduces for system (30) to the condition of phase flow invariance under the transformations $u \Rightarrow -u$, $v \Rightarrow -v$.

The stability of the singular points (31)–(34) may be found directly from the analysis of the linearized system (30). However, by virtue of continuous dependence of the roots of algebraic equation on its coefficients, we can state that the character and stability of the singular points (31)–(34) and (21)–(24) coincide for sufficiently small v_0. If there exists, for $v_0 = 0$, a structurally stable heteroclinic trajectory connecting the points P_1 and P_2 ($\overline{P_1}$ and $\overline{P_2}$) then, for small enough v_0, there exists a structurally stable heteroclinic trajectory corresponding to the intersection of two-dimensional stable and unstable manifolds of the singular points P_1' and P_2' ($\overline{P_1'}$ and $\overline{P_2'}$). In terms of the initial equation (1) this corresponds to a shock solution moving with constant velocity v_0 along the x-axis. The intersection of one-dimensional stable and unstable manifolds of the singular points P_1' and $\overline{P_2'}$ ($\overline{P_1'}$ and P_2') corresponds to hole solutions moving with constant velocity v_0. It is noteworthy, however, that the problem of structural stability of the heteroclinic trajectories corresponding to moving holes is more complicated than in the case $v_0 = 0$. The system of equations (30) is not invariant under the involutions (19) and (20). Consequently, the given heteroclinic trajectories should have co-dimension 1 even if the parameters Q_1 and Q_2 depend in a certain way on α and β. The existence of these trajectories at arbitrary values of α and β, as was ascertained in the works of Nozaki and Bekki [1], indicates that additional symmetries which are more complicated that those considered above are present in system (30). Thus, the results of Proposition 3 about multiloop trajectories can be carried over to the case $v_0 \neq 0$ only under the assumption of the existence of structurally stable heteroclinic trajectories connecting the singular points, supplemented by local analysis of phase flow in the neighbourhood of the saddle-focus. Such multiloop heteroclinic trajectories correspond to complex shock–hole–shock structures moving with constant velocity v_0.

5. Discussion

The results presented in this paper indicate that Eq. (1) has solutions describing complex structures, both moving and at rest, consisting of a sequence of holes and shocks. We would like to emphasize that we did not consider stability of these solutions. It is a nontrivial problem even for the solutions describing isolated holes (see, e.g., Refs. [4,7]).

Still another important issue is related to structural stability of the solutions of interest at weak disturbances in the right-hand side of system (1). This problem was analysed in Ref. [3] in applications to the isolated holes. It was shown that the family of holes disappears and only motionless structures are retained under perturbations of the form $d|s|^4 s$, $d \ll 1$ (the most typical in terms of the asymptotic theory). The fact that motionless structures are retained can also be explained in the context of the analysis of phase space structure presented above. Indeed, when the additional term $d|s|^4 s$ is introduced into the right-hand side of Eq. (1), system (12) retains the involutions (15) and (16). Consequently, at appropriate choice of the value of Q, the heteroclinic trajectories corresponding to holes will exist in a broad region of values of the parameters α and β (at least for $d \ll 1$). Note that these trajectories will no longer lie in a two-dimensional plane in three-dimensional space. As to the moving holes, we can only make a supposition that the symmetries considered in the previous section vanish at $d \neq 0$.

The same holds true for the solutions describing complex shock–hole–shock structures. The corresponding heteroclinic trajectories, that are structurally stable, exist at sufficiently small $d \neq 0$. It is significant that there may exist not only motionless but also moving structures of this type. Indeed, when proving the existence of the multiloop heteroclinic trajectories corresponding to these structures (Proposition 3) we showed that a finite (but not countable) number of such trajectories exists even when one-dimensional manifolds of singular points do not intersect. These intersections disappear at $d \neq 0$ and $v_0 \neq 0$. Whereas the intersection of two-dimensional stable and unstable manifolds is retained, as well as the structure of phase space in the neighbourhood of saddle-focus that was used in the proof of Proposition 3. Hence, a countable number of motionless and a finite number of moving structures of the shock–hole–shock type will exist for $d \neq 0$.

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